Improving the Finite-Length Performance of Spatially Coupled LDPC Codes by Connecting Multiple Code Chains

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Joint work with: R. Urbanke (EPFL), David G. M. Mitchell and Daniel J. Costello, Jr (Notre Dame University), Dmitri Truhachev (Dalhousie University)
A particularly exciting new class of LDPC codes

- Spatially-coupled LDPC (SC-LDPC) codes promise excellent performance over a broad range of channel conditions.
- Capacity approaching iterative decoding thresholds, characteristic of optimized irregular codes.
- Linear growth of minimum distance with block length, characteristic of regular codes.
Fig. 1. A sketch of typical LDPC-BC decoded BER performance over the AWGN channel. Also shown for comparison are the channel capacity, or Shannon limit, and the performance of uncoded binary phase-shift keying (BPSK) transmission.

In this paper, we highlight a particularly exciting new class of LDPC codes, called spatially-coupled LDPC (SC-LDPC) codes, which promise robustly excellent performance over a broad range of channel conditions, including both the waterfall and error floor regions of the BER curve. We also show how SC-LDPC codes can be viewed as a type of LDPC convolutional code (LDPC-CC), since spatial coupling is equivalent to introducing memory into the encoding process. In channel coding parlance, the key feature of SC-LDPC codes that distinguishes them from standard LDPC codes is their ability to combine the best features of regular and irregular codes in a single design: (1) capacity approaching iterative decoding thresholds, characteristic of optimized irregular codes, thus promising excellent performance in the waterfall, and (2) linear growth of minimum distance with block length, characteristic of regular codes, thus promising the elimination of an error floor. As will be discussed in more detail in Section II, this is achieved by introducing a slight structured irregularity into the Tanner graph representation of a regular LDPC code. An added feature of the SC-LDPC code design is that the resulting graph retains the essential implementation advantages associated with the structure of regular codes, compared to typical irregular designs. The research establishing the performance characteristics of SC-LDPC codes relies on ensemble average asymptotic methods, i.e., the capacity approaching thresholds and asymptotically good minimum distance behavior are shown to hold for typical members of SC-LDPC code ensembles as the block length tends to infinity. (Following the lead of Shannon, coding theorists often find it easier and more insightful to analyze the average asymptotic behavior of code ensembles than to determine the exact performance of specific codes.) These research results are summarized in Section II.

Section III discusses issues related to realizing the exceptional promise of SC-LDPC codes with specific code and decoder designs suitable for low-complexity implementation at block lengths typically employed in practice: 1) the use of high-throughput, parallel, pipeline decoding and 2) the use of sliding-window decoding strategies for reduced latency and computational complexity, and Section IV contains a short summary of several open research problems. Finally, Section V includes some concluding remarks along with references.

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SC-LDPC codes are known to have excellent asymptotic properties, but there are many open questions regarding code design for finite block lengths:

- Scaling the lifting factor $M$ and the termination length $L$ to achieve the best possible finite-length performance.
- Studying puncturing as a means of obtaining SC-LDPC codes with high rates.
- Analyzing and designing more powerful codes such as non-binary or generalized SC-LDPC codes.
In this talk ...

- A review of recent results to estimate the finite-length performance of SC-LDPC code chains in the waterfall region over the binary erasure channel (BEC).


- A novel encoding/transmission technique that improves the finite-length performance of a system using long SC-LDPC code chains.
Finite-length scaling behavior of \((l, r)\)-regular LDPC codes

Finite-length scaling behavior of SC-LDPC code chain ensembles

A closed SC-LDPC code chain

The Loop ensemble

Connecting consecutive chains
Finite-length analysis of binary LDPC codes is typically carried out over the BEC channel.

Reformulation of the BP decoder: peeling decoding.

Models capturing the dominant effects that relate error probability and code parameters can be proposed.

Scaling behavior identified over the BEC shows up in other channels.

\[ x_i \rightarrow (0, 1) \xrightarrow{1-\varepsilon} (0, 1) \]

Binary erasure channel
A convenient decoding algorithm: peeling decoding

- Equivalent to BP for the BEC case.
- Basic Iteration: one degree-one check node and one variable node are removed from the code graph.
  - No variable nodes: decoding succeeds.
  - No degree-one check nodes: decoding fails.

Finite-length LDPC performance prediction over the BEC

We study the statistical presence of degree-one check nodes as decoding evolves over time.

\[ r_1(\tau) = \frac{R_1(\tau)}{\varepsilon M}, \quad \tau = \frac{\text{Iteration index}}{\varepsilon M} \]
The (3, 6)-regular block code, $\varepsilon = 0.415$

Error probability is dominated by the zero-crossing probability at the local minima: critical point.
• For LDPC block codes, the evolution for both the mean $\hat{r}_1(\tau)$ and variance $\delta_1(\tau)$ of $r_1(\tau)$ can be analytically computed.

• System of differential equations related to the moments of the graph expected evolution in a single PD step:

\[
\frac{\partial \hat{r}_1(\tau)}{\partial \tau} = \mathbb{E}[\Delta_1(\tau)|\hat{D}\hat{D}(\tau)] = f(\Delta_1(\tau))
\]

\[
\frac{\partial \delta_1(\tau)}{\partial \tau} = \text{Var}[\Delta_1(\tau)|\hat{D}\hat{D}(\tau)] + \sum_{j=1}^{r} \delta_{1,j}(\tau) \frac{f(\Delta_j(\tau))}{r_1(\tau)} |_{\hat{D}\hat{D}(\tau)}
\]

where $\Delta_j(\tau) = R_{j}(\ell + 1) - R_{j}(\ell)$.
Furthermore, they showed that $r_1(\tau)$ converges (in $M$) to a Gaussian distribution and, at the critical point,

$$\frac{\hat{r}_1(\tau^*)}{\sqrt{\delta_1(\tau^*)}} = \frac{\sqrt{M}(\epsilon^* - \epsilon)}{\alpha} + O(M^{-1/6})$$

and, for $(l, r, \tau)$-regular LDPC block codes they gave analytic expression for $\alpha = \alpha(l, r)$.

$$P_{(l,r)} \approx Q\left(\frac{\sqrt{M}(\epsilon^* - \epsilon)}{\alpha}\right)$$
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The $C(3, 6, L)$ SC-LDPC code chain ensemble

Uncoupled ensemble

- $L$ uncoupled $(3, 6)$-regular LDPC block codes of length $M$.
- Each code contains $M$ variable nodes of degree 3 and $\frac{M}{2}$ check nodes of degree 6.

Coupling pattern

- The three edges of a variable node at position $u$ are connected to check nodes at positions $u - 1$, $u$, and $u + 1$.
- We terminate the code at each end of the chain with $\frac{M}{2}$ additional check nodes of degree 2.
- There are $D = L + 2$ occupied positions and the code length is $LM$. 
We approach capacity as $L$ grows.

\[ r_{C(3,6,L)} = \frac{1}{2} - O(L^{-1}) \]

\[ \varepsilon_{C(3,6,L)} \approx \varepsilon_{\text{MAP}}(3, 6) = 0.48815 \]

- White round nodes indicate positions where variable nodes are connected to some check nodes of degree lower than 6.
- Black round nodes indicate positions where variable nodes are connected to all check nodes of degree 6.
- Black square nodes indicate positions with no variable nodes.
Wave-like decoding, as BP iterates, information propagates from the boundary positions towards the middle positions.
A $\mathcal{C}(3, 6, 50)$ code chain under PD, $\varepsilon = 0.45$

$$r_1(\tau) = \frac{\sum_i \text{# degree one check nodes at position } i}{M}$$

We do not have a single critical time point at which the decoder is most likely to stop.

**Critical phase**

The mean and variance of $r_1(\tau)$ during the critical phase, remain essentially constant.
(l, r)-regular LDPC block code

\[
\frac{\partial \hat{r}_1(\tau)}{\partial \tau} = \mathbb{E}[\Delta_1(\tau) | \hat{D}D(\tau)] = f(\Delta_1(\tau))
\]

\[
\frac{\partial \delta_1(\tau)}{\partial \tau} = \text{Var}[\Delta_1(\tau) | \hat{D}D(\tau)] + \sum_{j=1}^r \delta_{1,j}(\tau) \frac{f(\Delta_j(\tau))}{r_1(\tau)} |_{\hat{D}D(\tau)}
\]

SC-LDPC

For different SC-LDPC code constructions, we have derived the differential equations to estimate the mean and variance evolution of the process:

- Random SC-LDPC codes, Olmos & Urbanke, 2014
- Protograph-based SC-LDPC codes Stinner & Olmos, 2014,
\( \hat{r}_1(\tau) \) for the \( C(3, 6, 50) \) chain

Olmos & Urbanke 2013:

\[
P_{C(3,6,L)} \approx 1 - \exp \left( - \frac{eL}{\mu_0(M,\epsilon,3,6)} \right)
\]

\( \mu_0(M,\epsilon,3,6) \): average survival time of the \( r_1(\tau) \) process

- \( \hat{r}_1(\tau) \) does not display a single critical point (as in the case of the uncoupled \((3, 6)\)-regular block code ensemble), but rather a critical phase of length \( \mathcal{O}(L) \) in which it remains constant.

\[
\mu_0(l, r, M) \propto \int_0^{\sqrt{M(\epsilon^*-\epsilon)}} e^{\frac{1}{2}z^2} \, dz
\]
Comparison with simulated performance. $C(3, 6, 50)$
\( \hat{r}_1(\tau) \) for the \( C(3, 6, 25) \) chain

- \( L = 25, L = 50 \)
- For short chains, the error probability is essentially described by a single-critical point (block code like) model!
- Short chains exhibit increased robustness compared to long chains (less time in a critical phase).
- For short (lower rate) chain lengths, the variable nodes at the central positions quickly benefit from the effect of the two stronger sub-codes at the chain boundaries.
Olmos, Mitchell, Truhachev, Costello 2013:

For short chain lengths...

\[ P_{C(3,6,L)} \approx Q \left( \frac{\sqrt{M}(\epsilon^* - \epsilon)}{\alpha} \right) \]

- Expression is independent of \( L \).

- Decoding does not explicitly exhibit wave-like behavior!

- There is an increase in decoding robustness beyond what is predicted by the long chain expression which depends on \( L \).

- Note that the critical point occurs at approximately same level as the critical phase.
Block error rate decay with the gap to threshold $\Delta \epsilon$

Ensembles $C(3, 6, 50)$ and $C(3, 6, 25)$ with $M = 1000$ bits per position

At moderate error rates, the scaling is more favorable for the short chain, $L = 25$.

Qualitative robustness improvement. The expression

$$1 - \exp \left( - \frac{\epsilon L}{\mu_0(M, \epsilon, 3, 6)} \right)$$

does not capture the right scaling for $L = 25$ (green dashed line)
• In the figure, we show the $\hat{r}_1(\tau)$ evolution for the $C(4, 8, L)$ code chain ensemble with $L = 25$ and $L = 50$.

• $\varepsilon_{C(4, 8, 50)} \approx \varepsilon_{\text{MAP}}(4, 8) = 0.4974$. 
• For short chains there exists an improvement in decoding robustness.

• However, the use of short lower-rate SC-LDPC code chains may not be a viable option in some applications.

• How could we improve the finite-length performance of a long SC-LDPC code chain?
Finite-length scaling behavior of \((l, r)\)-regular LDPC codes

Finite-length scaling behavior of SC-LDPC code chain ensembles

A closed SC-LDPC code chain

The Loop ensemble

Connecting consecutive chains
A closed SC-LDPC code chain

- By using the low-degree check nodes at the terminations, we can create variables of higher degree in intermediate positions of the chain (grey nodes).
- Maybe in this way the effect of the low-rate terminations is stronger along the chain.
\( \hat{\tau}_1(\tau) \) evolution.

\[
\begin{array}{c|c}
p & \varepsilon^* \\
\hline
9 & 0.467 \\
16 & 0.468 \\
22 & 0.472 \\
24 & 0.478 \\
25 & 0.488 \\
\end{array}
\]

Figure: \( \hat{\tau}_1(\tau) \) for \( \varepsilon = 0.47 \)

Either we do much worse than the standard \( C(3, 6, 50) \) code chain or we get similar performance.
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The loop connected chain ensemble

Improving Spatially Coupled LDPC Codes by Connecting Chains

Dmitri Truhachev*, David G. M. Mitchell†, Michael Lentmaier‡, and Daniel J. Costello, Jr.†

- Same rate than the standard SC-LDPC code chain.
- Double block length.
<table>
<thead>
<tr>
<th>Rate</th>
<th>Ensemble</th>
<th>$\varepsilon_{\mathcal{L}(3,6,L)}$</th>
<th>Ensemble</th>
<th>$\varepsilon_{\mathcal{C}(3,6,L)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.375</td>
<td>$\mathcal{L}(3,6,8)$</td>
<td>0.5445</td>
<td>$\mathcal{C}(3,6,8)$</td>
<td>0.522</td>
</tr>
<tr>
<td>0.4</td>
<td>$\mathcal{L}(3,6,10)$</td>
<td>0.5323</td>
<td>$\mathcal{C}(3,6,10)$</td>
<td>0.508</td>
</tr>
<tr>
<td>0.4167</td>
<td>$\mathcal{L}(3,6,12)$</td>
<td>0.5237</td>
<td>$\mathcal{C}(3,6,12)$</td>
<td>0.495</td>
</tr>
<tr>
<td>0.4333</td>
<td>$\mathcal{L}(3,6,15)$</td>
<td>0.5105</td>
<td>$\mathcal{C}(3,6,15)$</td>
<td>0.489</td>
</tr>
<tr>
<td>0.4444</td>
<td>$\mathcal{L}(3,6,18)$</td>
<td>0.4989</td>
<td>$\mathcal{C}(3,6,18)$</td>
<td>0.488</td>
</tr>
<tr>
<td>0.46</td>
<td>$\mathcal{L}(3,6,25)$</td>
<td>0.488</td>
<td>$\mathcal{C}(3,6,18)$</td>
<td>0.488</td>
</tr>
<tr>
<td>0.48</td>
<td>$\mathcal{L}(3,6,50)$</td>
<td>0.488</td>
<td>$\mathcal{C}(3,6,18)$</td>
<td>0.488</td>
</tr>
</tbody>
</table>

Table: BEC thresholds $\varepsilon^*$ for several SC-LDPC connected chain loop ensembles $\mathcal{L}(3,6,L)$ and single chain ensembles $\mathcal{C}(3,6,L)$. 
For low $L$ values, the loop ensemble achieves important gains in thresholds.

- Outer segments are strongly protected and, under the code threshold, are decoded with very high probability.
- The remaining ensemble looks as two independent chains of length $2L/3$.

E.g., for $L = 15$

By DE, we can verify that all variable nodes in outer segments are decoded up to an erasure probability of $\epsilon = 0.535$, while the code threshold is 0.5105.
For larger $L$ values, decoding relies on the low-rate terminations.

$$P_{\times L(l,r,L)} \rightarrow P_{2\times C(l,r,L)} \approx 1 - \exp \left( -2 \frac{\epsilon L}{\mu_0(M, \epsilon, l, r)} \right)$$

The loop ensemble will not improve the finite-length performance of long SC-LDPC code chains.
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New proposal: Continuous chain (CC) transmission of SC-LDPC code chains

- Consider a system of long chains in which the information stream is divided into blocks of $MLr_C(3,6,50)$ bits, which are then independently encoded, transmitted, and decoded at the receiver.
• The system performance can be improved if we do not transmit the chains independently.

• Instead, data is encoded in a continuous fashion using a convolutional-like structure based on connected SC-LDPC code chains.

• We refer to such an encoding scheme as continuous chain (CC) transmission of SC-LDPC codes.

• Decoding complexity and delay are not significantly affected.
Graph representation of CC transmission using the \( C(3, 6, 50) \) ensemble

- SC-LDPC code chains are not independently encoded and transmitted anymore.
- Chains are connected to increase the variable node degrees up to 5 for the nodes in the four middle positions of the previous chain (grey nodes).
- We create subregions of lower local rate.
No terminations with low degree-check nodes ...

but connected to two regions with lower local rate.

By DE, we can verify that all variable nodes in grey regions are decoded up to an erasure probability of $\epsilon = 0.505$, while the code threshold is 0.48815.
Hence, if $\varepsilon \leq \varepsilon_{C(3,6,50)} \approx 0.48815$, the grey regions will be decoded almost surely ...

For $N$ consecutive chains encoded in this way, chains 1 to $(N - 1)$ are broken into two shorter chains after grey regions are decoded.

Better finite-length decoding properties!

We don’t lose any rate!
The three layer case \( N = 3 \)

- The performance in the last chain (dashed lines) is similar to that of a single chain.
- We obtain a significant gain in performance for the first two chains (solid lines). Unequal error protection!
Feasibility of CC transmission

- No significant increase in decoding complexity.
- No significant increase in decoding delay using a sliding window decoder.
- Only requires a small additional memory and a different transmission order for the encoded bits.

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- Details on the feasibility of CC transmission.
- Alternative constructions.
- Results for the binary-input AWGN channel.